

Max- Relative Entropy of Entanglement, *alias* Log Robustness

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Properties of the *max- relative entropy of entanglement*, defined in [10], are investigated, and its significance as an upper bound to the one-shot rate for *perfect* entanglement dilution, under a particular class of quantum operations, is discussed. It is shown that it is in fact equal to another known entanglement monotone, namely the *log robustness*, defined in [7]. It is known that the latter is not asymptotically continuous and it is not known whether it is weakly additive. However, by suitably modifying the max- relative entropy of entanglement we obtain a quantity which is seen to satisfy both these properties. In fact, the modified quantity is shown to be equal to the regularised relative entropy of entanglement.

INTRODUCTION

In [23], Renner introduced two important entropic quantities, called the min- and max- entropies. Recently, the operational meanings of these quantities, i.e., their relevance with regard to actual information-processing tasks, was elucidated in [15]. Further, two new relative entropy quantities, which act as parent quantities for these min- and max- entropies, were introduced in [10]. These were, namely, the max-relative entropy, $D_{\max}(\rho||\sigma)$, and the min-relative entropy, $D_{\min}(\rho||\sigma)$. Here ρ denotes a state and σ denotes a positive operator. Various properties of these quantities were proved in [10]. In particular, it was shown that

$$D_{\min}(\rho||\sigma) \leq S(\rho||\sigma) \leq D_{\max}(\rho||\sigma),$$

where $S(\rho||\sigma)$ is the relative entropy of ρ and σ .

In addition, it was shown in [10] that the minimum over all separable states, σ , of $D_{\max}(\rho||\sigma)$, defines a (full) entanglement monotone [33] for a bipartite state ρ . This quantity, referred to as the *max-relative entropy of entanglement* and denoted by $E_{\max}(\rho)$, was proven to be an upper bound to the relative entropy of entanglement, $E_R(\rho)$ [29].

In this paper we investigate further properties of $E_{\max}(\rho)$ and discuss its significance. We prove that it is quasiconvex, i.e., for a mixture of states $\rho = \sum_{i=1}^n p_i \rho_i$, $E_{\max}(\rho) \leq \max_{1 \leq i \leq n} E_{\max}(\rho_i)$. We also infer that it is not asymptotically continuous [12], and does not reduce to the *entropy of entanglement* for pure bipartite states (that is, to the entropy of the reduced state of either of the two parties). We do so by showing that $E_{\max}(\rho)$ is in fact equal to another known entanglement monotone, namely, the *log robustness* [7]: $LR_g(\rho) := \log(1 + R_g(\rho))$. Here $R_g(\rho)$ denotes the global robustness [13] of ρ , which is a measure of the amount of noise that can be added to an entangled state ρ before it becomes unentangled (separable). By suitably modifying $E_{\max}(\rho)$, we arrive at a quantity, which we denote by $\mathcal{E}_{\max}(\rho)$, and which is asymptotically continuous and weakly additive [34]. Asymptotic continuity (24) is proved by showing that

$\mathcal{E}_{\max}(\rho)$ is equal to the regularised relative entropy of entanglement $E_R^\infty(\rho)$ [1], for which this property has been proved [9]. The necessary modifications involve (i) “smoothing” $E_{\max}(\rho)$ to obtain the *smooth max-relative entropy of entanglement* $E_{\max}^\varepsilon(\rho)$, for any fixed $\varepsilon > 0$. (This is similar to the smoothing introduced by Renner [23] to obtain the smooth Rényi entropies from the min- and max- entropies mentioned above); (ii) regularising, and (iii) taking the limit $\varepsilon \rightarrow 0$ (see the following sections).

It would be natural to proceed analogously with the min-relative entropy and define a quantity, $E_{\min}(\rho)$, to be the minimum over all separable states, σ , of $D_{\min}(\rho||\sigma)$. However, it can be shown [35] that $E_{\min}(\rho)$ is not a full entanglement monotone. It *can* increase on average under local operations and classical communication (LOCC). Instead $E_{\min}(\rho)$ satisfies a weaker condition of monotonicity under LOCC maps, that is, $E_{\min}(\rho) \geq E_{\min}(\Lambda(\rho))$, for any LOCC operation Λ . Nevertheless, as for the case of $E_{\max}(\rho)$, a “smoothing” of $E_{\min}(\rho)$, followed by regularisation, yields a quantity which is equal to $E_R^\infty(\rho)$, in the limit of the smoothing parameter $\varepsilon \rightarrow 0$. This will be presented in a forthcoming paper [6].

In [7] it was shown that $E_R^\infty(\rho)$ is equal to both the entanglement cost and the distillable entanglement under the set of quantum operations which do not generate any entanglement asymptotically [for details, see [7]]. This gives an operational significance to the regularised version of the smooth max- and min- relative entropies of entanglement, in the limit $\varepsilon \rightarrow 0$.

For a given $\varepsilon > 0$, the smoothed versions, $E_{\max}^\varepsilon(\rho)$ and $E_{\min}^\varepsilon(\rho)$, of the max- and min- relative entropies of entanglement, also have operational interpretations. They arise as optimal rates of entanglement manipulation protocols involving separability-preserving maps. A quantum operation Λ is said to be a separability-preserving map if $\Lambda(\sigma)$ is separable for any separable state σ . These maps constitute the largest class of operations which cannot create entanglement and contains the class of separable operations [2, 20, 29]. [See [7] for details]. The quantities $E_{\max}^\varepsilon(\rho)$ and $E_{\min}^\varepsilon(\rho)$ can be interpreted as one-shot rates of entanglement dilution and distillation protocols

involving separability-preserving maps, for a given bound on the corresponding probabilities of error, i.e., when the probability of error associated with the protocol is bounded above by the smoothing parameter ε . This is analogous to the interpretation of the ε -smooth Rényi entropies, as one-shot rates of various protocols [23, 25, 27], when the probability of error is at most ε . Evaluation of these one-shot rates for the entanglement manipulation protocols, will be presented in a forthcoming paper [6].

The max-relative entropy of entanglement (or log robustness), $E_{\max}(\rho)$, provides an upper bound to the one-shot *perfect* entanglement cost, not under LOCC maps, but under quantum operations which generate an entanglement (as measured by the global robustness) of at most $1/R_g(\rho)$. We shall refer to such maps as α_ρ -separability preserving (or α_ρ -SEPP) maps, with $\alpha_\rho = 1/R_g(\rho)$. This is elaborated below.

We start the main body of our paper with some mathematical preliminaries. Next we recall the definitions of the relevant relative entropy quantities and entanglement monotones, and prove that $E_{\max}(\rho)$ is quasiconvex. We then show that it is equal to the global log robustness, and that it does not in general reduce to the relative entropy of entanglement for pure states. Next we define the smooth max-relative entropy of entanglement and $\mathcal{E}_{\max}(\rho)$, and prove that the latter is weakly additive. Our main result is given in Theorem 1, which states that $\mathcal{E}_{\max}(\rho) = E_R^\infty(\rho)$ [36].

MATHEMATICAL PRELIMINARIES

Let $\mathcal{B}(\mathcal{H})$ denote the algebra of linear operators acting on a finite-dimensional Hilbert space \mathcal{H} . The von Neumann entropy of a state ρ , i.e., a positive operator of unit trace in $\mathcal{B}(\mathcal{H})$, is given by $S(\rho) = -\text{Tr} \rho \log \rho$. Throughout this paper, we take the logarithm to base 2 and all Hilbert spaces considered are finite-dimensional. In fact, since in this paper we consider bipartite states, the underlying Hilbert space is given by $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. Let \mathcal{D} denote the set of states in $\mathcal{B}(\mathcal{H})$, and let $\mathcal{S} \subset \mathcal{D}$ denote the set of separable states. Further, let \mathcal{S}_n denote the set of separable states in $\mathcal{B}(\mathcal{H}^{\otimes n})$.

The trace distance between two operators A and B is given by

$$\|A - B\|_1 := \text{Tr}[\{A \geq B\}(A - B)] - \text{Tr}[\{A < B\}(A - B)] \quad (1)$$

The fidelity of states ρ and ρ' is defined to be

$$F(\rho, \rho') := \text{Tr} \sqrt{\rho^{\frac{1}{2}} \rho' \rho^{\frac{1}{2}}}.$$

The trace distance between two states ρ and ρ' is related to the fidelity $F(\rho, \rho')$ as follows (see (9.110) of [17]):

$$\frac{1}{2} \|\rho - \rho'\|_1 \leq \sqrt{1 - F(\rho, \rho')^2} \leq \sqrt{2(1 - F(\rho, \rho'))}. \quad (2)$$

We also use the “gentle measurement” lemma [19, 32]:

Lemma 1 *For a state ρ and operator $0 \leq \Lambda \leq I$, if $\text{Tr}(\rho\Lambda) \geq 1 - \delta$, then*

$$\|\rho - \sqrt{\Lambda} \rho \sqrt{\Lambda}\|_1 \leq 2\sqrt{\delta}.$$

The same holds if ρ is only a subnormalized density operator.

DEFINITIONS OF MIN- AND MAX- RELATIVE ENTROPIES

Definition 1 *The max- relative entropy of a state ρ and a positive operator σ is given by*

$$D_{\max}(\rho||\sigma) := \log \min\{\lambda : \rho \leq \lambda\sigma\} \quad (3)$$

Note that $D_{\max}(\rho||\sigma)$ is well-defined if $\text{supp } \rho \subseteq \text{supp } \sigma$.

Definition 2 *The min- relative entropy of a state ρ and a positive operator σ is given by*

$$D_{\min}(\rho||\sigma) := -\log \text{Tr}(\pi\sigma), \quad (4)$$

where π denotes the projector onto $\text{supp } \rho$, the support of ρ . It is well-defined if $\text{supp } \rho$ has non-zero intersection with $\text{supp } \sigma$.

Various properties of $D_{\min}(\rho||\sigma)$ and $D_{\max}(\rho||\sigma)$ were proved in [10]. In this paper we shall use the following properties of the max- relative entropy, $D_{\max}(\rho||\sigma)$:

- The max- relative entropy is monotonic under completely positive trace-preserving (CPTP) maps, i.e., for a state ρ , a positive operator σ , and a CPTP map Λ :

$$D_{\max}(\Lambda(\rho)||\Lambda(\sigma)) \leq D_{\max}(\rho||\sigma) \quad (5)$$

- The max- relative entropy is quasiconvex, i.e., for two mixtures of states, $\rho := \sum_{i=1}^n p_i \rho_i$ and $\omega := \sum_{i=1}^n p_i \omega_i$,

$$D_{\max}(\rho||\omega) \leq \max_{1 \leq i \leq n} D_{\max}(\rho_i||\omega_i). \quad (6)$$

- $D_{\max}(\rho \otimes \rho||\omega \otimes \omega) = 2D_{\max}(\rho||\omega)$. This property follows directly from the definition (3).

The min- and max- (unconditional and conditional) entropies, introduced by Renner in [23] are obtained from $D_{\min}(\rho||\sigma)$ and $D_{\max}(\rho||\sigma)$ by making suitable substitutions for the positive operator σ (see [10] for details).

SMOOTH MIN- AND MAX- RELATIVE ENTROPIES

Smooth min- and max- relative entropies are generalizations of the above-mentioned relative entropy measures, involving an additional *smoothness* parameter $\varepsilon \geq 0$. For $\varepsilon = 0$, they reduce to the *non-smooth* quantities.

Definition 3 For any $\varepsilon \geq 0$, the ε -smooth min- and max-relative entropies of a bipartite state ρ relative to a state σ are defined by

$$D_{\min}^{\varepsilon}(\rho||\sigma) := \max_{\bar{\rho} \in B^{\varepsilon}(\rho)} D_{\min}(\bar{\rho}||\sigma)$$

and

$$D_{\max}^{\varepsilon}(\rho||\sigma) := \min_{\bar{\rho} \in B^{\varepsilon}(\rho)} D_{\max}(\bar{\rho}||\sigma) \quad (7)$$

where $B^{\varepsilon}(\rho) := \{\bar{\rho} \geq 0 : \|\bar{\rho} - \rho\|_1 \leq \varepsilon, \text{Tr}(\bar{\rho}) \leq \text{Tr}(\rho)\}$.

The following two lemmas are used to prove our main result, Theorem 1.

Lemma 2 Let ρ_{AB} and σ_{AB} be density operators, let Δ_{AB} be a positive operator, and let $\lambda \in \mathbb{R}$ such that

$$\rho_{AB} \leq 2^{\lambda} \cdot \sigma_{AB} + \Delta_{AB}.$$

Then $D_{\max}^{\varepsilon}(\rho_{AB}||\sigma_{AB}) \leq \lambda$ for any $\varepsilon \geq \sqrt{8\text{Tr}(\Delta_{AB})}$.

Lemma 3 Let ρ_{AB} and σ_{AB} be density operators. Then

$$D_{\max}^{\varepsilon}(\rho_{AB}||\sigma_{AB}) \leq \lambda$$

for any $\lambda \in \mathbb{R}$ and

$$\varepsilon = \sqrt{8\text{Tr}[\{\rho_{AB} > 2^{\lambda}\sigma_{AB}\}\rho_{AB}]}.$$

The proofs of these lemmas are analogous to the proofs of Lemmas 5 and 6 of [11] and are given in the Appendix for completeness.

THE MAX-RELATIVE ENTROPY OF ENTANGLEMENT

For a bipartite state ρ , the max-relative entropy of entanglement [10] is given by

$$E_{\max}(\rho) := \min_{\sigma \in \mathcal{S}} D_{\max}(\rho||\sigma), \quad (8)$$

where the minimum is taken over the set \mathcal{S} of all separable states.

It was proved in [10] that

$$E_{\max}(\rho) \geq E_R(\rho), \quad (9)$$

where $E_R(\rho) := \min_{\sigma \in \mathcal{S}} S(\rho||\sigma)$, the *relative entropy of entanglement* of the state ρ .

That $E_{\max}(\rho)$ is a full entanglement monotone follows from the fact that $D_{\max}(\rho||\sigma)$ satisfies a set of sufficient criteria [29] which ensure that $E_{\max}(\rho)$ has the following properties: (a) it vanishes if and only if ρ is separable, (b) it is left invariant by local unitary operations and (c) it does not increase on average under LOCC. This was proved in [10].

Lemma 4 The max-relative entropy of entanglement $E_{\max}(\rho)$ is quasiconvex, i.e., for a mixture of states $\rho = \sum_{i=1}^n p_i \rho_i$,

$$E_{\max}(\rho) \leq \max_{1 \leq i \leq n} E_{\max}(\rho_i). \quad (10)$$

Proof For each state ρ , let σ_{ρ} be a separable state for which

$$E_{\max}(\rho) = D_{\max}(\rho||\sigma_{\rho}).$$

Since the set of separable states is convex, and the max-relative entropy is quasiconvex (6), we have

$$\begin{aligned} E_{\max}\left(\sum_i p_i \rho_i\right) &\leq D_{\max}\left(\sum_i p_i \rho_i || \sum_i p_i \sigma_{\rho_i}\right) \\ &\leq \max_i D_{\max}(\rho_i || \sigma_{\rho_i}) \\ &= \max_i E_{\max}(\rho_i) \end{aligned} \quad (11)$$

■

Since $E_{\max}(\rho)$ is given by a minimisation over separable states, it is subadditive. Let σ be a separable state for which $E_{\max}(\rho) = D_{\max}(\rho||\sigma)$. Then,

$$\begin{aligned} E_{\max}(\rho \otimes \rho) &= \min_{\omega \in \mathcal{S}_2} D_{\max}(\rho \otimes \rho || \omega) \\ &\leq D_{\max}(\rho \otimes \rho || \sigma \otimes \sigma) \\ &= 2D_{\max}(\rho||\sigma) = 2E_{\max}(\rho). \end{aligned} \quad (12)$$

Lemma 5 The max-relative entropy of entanglement $E_{\max}(\rho)$ of a bipartite state ρ is equal to its global log robustness of entanglement [7], which is defined as follows:

$$LR_g(\rho) := \log(1 + R_g(\rho)), \quad (13)$$

where $R_g(\rho)$ is the global robustness of entanglement [13], given by

$$R_g(\rho) = \min_{s \in \mathbb{R}} \left\{ s \geq 0 : \exists \omega \in \mathcal{D} \text{ s.t. } \frac{1}{1+s}\rho + \frac{s}{1+s}\omega \in \mathcal{S} \right\}$$

Proof We can equivalently write $R_g(\rho)$ as follows:

$$\begin{aligned} R_g(\rho) &= \min_{s \in \mathbb{R}} \left\{ s \geq 0 : \exists \omega \in \mathcal{D} \text{ s.t. } \rho + s\omega = (1+s)\sigma, \sigma \in \mathcal{S} \right\} \\ &= \min_{s \in \mathbb{R}} \left\{ s \geq 0 : \exists \sigma \in \mathcal{S} \text{ s.t. } \rho \leq (1+s)\sigma \right\}, \end{aligned} \quad (14)$$

since, defining $\tilde{\omega} := (1 + s)\sigma - \rho$, we see that $\text{Tr} \tilde{\omega} = 1 + s - 1 = s$, hence allowing us to write $\tilde{\omega} = s\omega$ for some $\omega \in \mathcal{D}$. Hence,

$$\log(1 + R_g(\rho)) = \min_{\sigma \in \mathcal{S}} D_{\max}(\rho || \sigma).$$

■

Definition 4 A state π for which

$$\rho + R_g(\rho)\pi = (1 + R_g(\rho))\sigma,$$

for some separable state σ , is referred to as an *optimal state* for ρ in the global robustness of entanglement.

It was shown in [13] that for a pure bipartite state $\rho = |\psi\rangle\langle\psi| \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$,

$$R_g(\rho) = \left(\sum_{i=1}^m \lambda_i \right)^2 - 1,$$

where the λ_i , $i = 1, \dots, m$, denote the Schmidt coefficients of $|\psi\rangle$. This implies that for the pure state $\rho = |\psi\rangle\langle\psi|$, the max-relative entropy of entanglement is given by

$$E_{\max}(\rho) = \log(1 + R_g(\rho)) = 2 \log \left(\sum_{i=1}^m \lambda_i \right), \quad (15)$$

i.e., twice the logarithm of the sum of the square roots of the eigenvalues of the reduced density matrix $\rho_\psi^A := \text{Tr}_B |\psi\rangle\langle\psi|$. Hence for a pure state ρ , $E_{\max}(\rho)$ does not in general reduce to its *entropy of entanglement* (i.e., the von Neumann entropy of ρ_ψ^A), even though it does so for a maximally entangled state. Let $\Psi_M \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$ denote a maximally entangled state (MES) of rank M , i.e., $\Psi_M = |\Psi_M\rangle\langle\Psi_M|$, with

$$|\Psi_M\rangle = \frac{1}{\sqrt{M}} \sum_{i=1}^M |i\rangle|i\rangle.$$

Then,

$$E_{\max}(\Psi_M) = \log M = S(\text{Tr}_A \Psi_M).$$

Moreover, $R_g(\Psi_M) = M - 1$.

Note that the right hand side of (15) is equal to the expression for another known entanglement monotone, namely the *logarithmic negativity* [21]

$$LN(\rho) := \log \|\rho^\Gamma\|_1,$$

for the pure state $\rho = |\psi\rangle\langle\psi|$. Here ρ^Γ denotes the partial transpose with respect to the subsystem A , and $\|\omega\|_1 = \text{Tr} \sqrt{\omega^\dagger \omega}$. It is known that $LN(\rho)$ is additive [21] for pure states, and we therefore have

$$E_{\max}(|\psi\rangle\langle\psi| \otimes |\phi\rangle\langle\phi|) = E_{\max}(|\psi\rangle\langle\psi|) + E_{\max}(|\phi\rangle\langle\phi|). \quad (16)$$

This additivity relation does not extend to mixed states in general. However, the following relation can be proved to hold [6]:

$$\begin{aligned} E_{\max}(\rho \otimes \Psi_M) &= E_{\max}(\rho) + E_{\max}(\Psi_M) \\ &= E_{\max}(\rho) + \log M. \end{aligned} \quad (17)$$

As mentioned in the Introduction $E_{\max}(\rho)$ provides an upper bound to the one-shot *perfect* entanglement cost, under quantum operations which generate an entanglement of at most $R_g(\rho)$. This is elaborated below.

In entanglement dilution the aim is to obtain a state ρ from a maximally entangled state. This cannot necessarily be done by using a single copy of the maximally entangled state and acting on it by a LOCC map. However, a single perfect copy of ρ can be obtained from a single copy of a maximally entangled state if one does not restrict the quantum operation employed to be a LOCC map but instead allows quantum operations which generate an entanglement of at most $1/R_g(\rho)$. Before proving this, let us state the definition of one-shot perfect entanglement cost of a state under a quantum operation Λ .

Definition 5 A real number R is said to be an achievable one-shot perfect dilution rate, for a state ρ , under a quantum operation Λ , if $\Lambda(\Psi_M) = \rho$ and $\log M \leq R$.

Definition 6 The one-shot perfect entanglement cost of a state under a quantum operation Λ is given by $E_{c,\lambda}^{(1)} = \inf R$, where the infimum is taken over all achievable rates.

Consider the quantum operation Λ which acts on any state ω as follows:

$$\Lambda_M(\omega) = \text{Tr}(\Psi_M \omega) \rho + (1 - \text{Tr}(\Psi_M \omega)) \pi, \quad (18)$$

where π is an optimal state for ρ in the global robustness of entanglement. It was shown in [7] that for $M = 1 + s$, where $s = R_g(\rho)$, the quantum operation Λ_M is an $(1/s)$ -separability preserving map (SEPP), i.e., for any separable state σ :

$$R_g(\Lambda(\sigma)) \leq 1/s.$$

In other words, the map Λ as defined by (18), is a quantum operation which generates an entanglement corresponding to a global robustness of at most $1/R_g(\rho)$.

Now if $\omega = \Psi_M$, then $\Lambda(\omega) = \rho$, and hence a perfect copy of ρ is obtained from a single copy of the maximally entangled state Ψ_M . The associated rate, R , of the one-shot entanglement dilution protocol corresponding to the map Λ_M satisfies the bound [37]:

$$R \leq \log M = \log(1 + s) = E_{\max}(\rho). \quad (19)$$

The log robustness, $LR_g(\rho)$, is not asymptotically continuous [7] and it is not known whether it is weakly additive. However, as mentioned in the Introduction, by

suitable modifying $E_{\max}(\rho)$ one can arrive at a quantity which is both asymptotically continuous and weakly additive. The necessary modifications involve (i) “smoothing” $E_{\max}(\rho)$ to obtain the *smooth max-relative entropy of entanglement* $E_{\max}^\varepsilon(\rho)$, for any fixed $\varepsilon > 0$; (ii) regularising, and (iii) taking the limit $\varepsilon \rightarrow 0$, as described below.

SMOOTH MAX-RELATIVE ENTROPY OF ENTANGLEMENT AND $\mathcal{E}_{\max}(\rho)$

For any $\varepsilon > 0$, we define the *smooth max-relative entropy of entanglement* of a bipartite state ρ , as follows:

$$\begin{aligned} E_{\max}^\varepsilon(\rho) &:= \min_{\bar{\rho} \in B^\varepsilon(\rho)} E_{\max}(\bar{\rho}) \\ &= \min_{\bar{\rho} \in B^\varepsilon(\rho)} \min_{\sigma \in \mathcal{S}} D_{\max}(\bar{\rho} \parallel \sigma), \\ &= \min_{\sigma \in \mathcal{S}} D_{\max}^\varepsilon(\rho \parallel \sigma), \end{aligned} \quad (20)$$

where $D_{\max}^\varepsilon(\rho \parallel \sigma)$ is the smooth max-relative entropy defined by (7). Further, we define its regularised version

$$\mathcal{E}_{\max}^\varepsilon(\rho) := \limsup_{n \rightarrow \infty} \frac{1}{n} E_{\max}^\varepsilon(\rho^{\otimes n}), \quad (21)$$

and the quantity

$$\mathcal{E}_{\max}(\rho) := \lim_{\varepsilon \rightarrow 0} \mathcal{E}_{\max}^\varepsilon(\rho) \quad (22)$$

Lemma 6 *The quantity $\mathcal{E}_{\max}(\rho)$ characterizing a bipartite state $\rho \in \mathcal{B}(\mathcal{H})$ and defined by (22), satisfies the following properties:*

1. *It is weakly additive, i.e., for any positive integer m ,*

$$\mathcal{E}_{\max}(\rho^{\otimes m}) = m \mathcal{E}_{\max}(\rho). \quad (23)$$

2. *It is asymptotically continuous, i.e., for a given $\varepsilon > 0$, if $\rho_m \in \mathcal{B}(\mathcal{H}^{\otimes m})$ is an operator for which $\|\rho_m - \rho^{\otimes m}\|_1 \leq \varepsilon$, then*

$$\left| \frac{\mathcal{E}_{\max}(\rho_m) - \mathcal{E}_{\max}(\rho^{\otimes m})}{m} \right| \leq f(\varepsilon), \quad (24)$$

where $f(\varepsilon)$ is a real function of ε such that $f(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$.

Proof Here we give the proof of 1, by showing that $\mathcal{E}_{\max}(\rho \otimes \rho) = 2\mathcal{E}_{\max}(\rho)$. The proof of 2 follows from Theorem 1 below since the regularized relative entropy of entanglement $E_R^\infty(\rho)$, (defined by (37)), is known to be asymptotically continuous [9].

We first prove that

$$\mathcal{E}_{\max}(\rho \otimes \rho) \leq 2\mathcal{E}_{\max}(\rho) \quad (25)$$

Fix $\varepsilon > 0$. Then,

$$\begin{aligned} E_{\max}^\varepsilon(\rho^{\otimes n}) &= \min_{\sigma_n \in \mathcal{S}_n} D_{\max}^\varepsilon(\rho^{\otimes n} \parallel \sigma_n), \\ &= \min_{\sigma_n \in \mathcal{S}_n} \min_{\bar{\rho}_n \in B^\varepsilon(\rho^{\otimes n})} D_{\max}(\bar{\rho}_n \parallel \sigma_n) \end{aligned} \quad (26)$$

$$= D_{\max}(\rho_n^\varepsilon \parallel \sigma_n^\varepsilon), \quad (27)$$

where $\rho_n^\varepsilon \in B^\varepsilon(\rho^{\otimes n})$ and $\sigma_n^\varepsilon \in \mathcal{S}_n$ are operators for which the minima in (26) are achieved.

Since $\rho_n^\varepsilon \in B^\varepsilon(\rho^{\otimes n})$, we have that $\|\rho_n^\varepsilon - \rho^{\otimes n}\|_1 \leq \varepsilon$, which in turn implies that

$$\|\rho_n^\varepsilon \otimes \rho_n^\varepsilon - \rho^{\otimes 2n}\|_1 \leq 2\varepsilon.$$

Therefore, $\rho_n^\varepsilon \otimes \rho_n^\varepsilon \in B^\varepsilon(\rho^{\otimes 2n})$. Further, since $\sigma_n^\varepsilon \otimes \sigma_n^\varepsilon \in \mathcal{S}_{2n}$, we have

$$\begin{aligned} E_{\max}^{2\varepsilon}(\rho^{\otimes 2n}) &= \min_{\bar{\rho}_{2n} \in B^{2\varepsilon}(\rho^{\otimes 2n})} \min_{\sigma_{2n} \in \mathcal{S}_{2n}} D_{\max}(\bar{\rho}_{2n} \parallel \sigma_{2n}) \\ &\leq D_{\max}(\rho_n^\varepsilon \otimes \rho_n^\varepsilon \parallel \sigma_n^\varepsilon \otimes \sigma_n^\varepsilon) \\ &= 2D_{\max}(\rho_n^\varepsilon \parallel \sigma_n^\varepsilon) \\ &= 2E_{\max}^\varepsilon(\rho^{\otimes n}). \end{aligned} \quad (28)$$

Hence,

$$\limsup_{n \rightarrow \infty} \frac{1}{n} E_{\max}^{2\varepsilon}((\rho \otimes \rho)^{\otimes n}) \leq 2 \limsup_{n \rightarrow \infty} \frac{1}{n} E_{\max}^\varepsilon(\rho^{\otimes n}), \quad (29)$$

that is, $\mathcal{E}_{\max}^{2\varepsilon}(\rho \otimes \rho) \leq 2\mathcal{E}_{\max}^\varepsilon(\rho)$. Taking the limit $\varepsilon \rightarrow 0$ on either side of this inequality yields the desired bound (25).

In fact, the identity holds in (25). This is simply because

$$\begin{aligned} \mathcal{E}_{\max}^\varepsilon(\rho \otimes \rho) &= \limsup_{n \rightarrow \infty} \frac{1}{n} E_{\max}^\varepsilon((\rho \otimes \rho)^{\otimes n}), \\ &= 2 \limsup_{n \rightarrow \infty} \frac{1}{2n} E_{\max}^\varepsilon((\rho)^{\otimes 2n}), \\ &= 2\mathcal{E}_{\max}^\varepsilon(\rho) \end{aligned} \quad (30)$$

The last line of (30) is proved [8] by employing the monotonicity (5) of the max- relative entropy under partial trace. We know that

$$\begin{aligned} \mathcal{E}_{\max}^\varepsilon(\rho) &:= \limsup_{n \rightarrow \infty} \frac{1}{n} E_{\max}^\varepsilon((\rho)^{\otimes n}), \\ &\geq \limsup_{n \rightarrow \infty} \frac{1}{2n} E_{\max}^\varepsilon((\rho)^{\otimes 2n}), \end{aligned} \quad (31)$$

However, it can be proven that the identity always holds in (31). This is done by assuming that

$$\mathcal{E}_{\max}^\varepsilon(\rho) > \limsup_{n \rightarrow \infty} \frac{1}{2n} E_{\max}^\varepsilon((\rho)^{\otimes 2n}), \quad (32)$$

and showing that this leads to a contradiction.

The assumption (32) implies that there exists a sequence of odd integers n_i for which

$$\limsup_{n_i \rightarrow \infty} \frac{1}{n_i} E_{\max}^\varepsilon(\rho^{\otimes n_i}) > \limsup_{n_i \rightarrow \infty} \frac{1}{n_i + 1} E_{\max}^\varepsilon(\rho^{\otimes n_i + 1}). \quad (33)$$

Let $\rho_{n_i+1} \in \mathcal{H}^{\otimes n_i+1}$ be an operator in $B^\varepsilon(\rho^{\otimes n_i+1})$ for which $E_{\max}^\varepsilon(\rho^{\otimes n_i+1}) = E_{\max}(\rho_{n_i+1})$. Then, using the monotonicity (5) of the max-relative entropy under partial trace, we have

$$\begin{aligned} E_{\max}^\varepsilon(\rho^{\otimes n_i+1}) &= E_{\max}(\rho_{n_i+1}) \\ &\geq E_{\max}(\text{Tr}_{\mathcal{H}}(\rho_{n_i+1})) \\ &\geq E_{\max}^\varepsilon(\rho^{\otimes n_i}), \end{aligned} \quad (34)$$

since $\text{Tr}_{\mathcal{H}}(\rho_{n_i+1}) \in B^\varepsilon(\rho^{\otimes n_i})$. Therefore,

$$\begin{aligned} \limsup_{n_i \rightarrow \infty} \frac{1}{n_i} E_{\max}^\varepsilon(\rho^{\otimes n_i}) &> \limsup_{n_i \rightarrow \infty} \frac{1}{n_i+1} E_{\max}^\varepsilon(\rho^{\otimes n_i+1}) \\ &\geq \limsup_{n_i \rightarrow \infty} \frac{1}{n_i+1} E_{\max}^\varepsilon(\rho^{\otimes n_i}) \\ &= \limsup_{n_i \rightarrow \infty} \frac{1}{n_i} E_{\max}^\varepsilon(\rho^{\otimes n_i}), \end{aligned} \quad (35)$$

which is a contradiction. \blacksquare

MAIN RESULT

Our main result is given by the following theorem.

Theorem 1

$$\mathcal{E}_{\max}(\rho) = E_R^\infty(\rho), \quad (36)$$

where, $E_R^\infty(\rho)$ denotes the regularized relative entropy of entanglement:

$$E_R^\infty(\rho) := \lim_{n \rightarrow \infty} \frac{1}{n} E_R(\rho^{\otimes n}), \quad (37)$$

where, $E_R(\rho) = \min_{\sigma \in \mathcal{S}} S(\rho||\sigma)$ is the relative entropy of entanglement of ρ .

Proof We first prove that $E_R^\infty(\rho) \leq \mathcal{E}_{\max}(\rho)$.

Fix $\varepsilon > 0$.

$$E_{\max}^\varepsilon(\rho^{\otimes n}) = \min_{\sigma_n \in \mathcal{S}_n} D_{\max}^\varepsilon(\rho^{\otimes n}||\sigma_n), \quad (38)$$

where \mathcal{S}_n denotes the set of separable states in $\mathcal{B}(\mathcal{H}^{\otimes n})$. In the above,

$$D_{\max}^\varepsilon(\rho^{\otimes n}||\sigma_n) = \min_{\bar{\rho}_n \in B^\varepsilon(\rho^{\otimes n})} D_{\max}(\bar{\rho}_n||\sigma_n) \quad (39)$$

Let $\rho_n^\varepsilon \in B^\varepsilon(\rho^{\otimes n})$ be the operator for which the minimum is achieved in (39). Hence,

$$E_{\max}^\varepsilon(\rho^{\otimes n}) = \min_{\sigma_n \in \mathcal{S}_n} D_{\max}(\rho_n^\varepsilon||\sigma_n) \quad (40)$$

Further, let $\tilde{\sigma}_n$ be the separable state for which the minimum is achieved in (40). Hence,

$$E_{\max}^\varepsilon(\rho^{\otimes n}) = D_{\max}(\rho_n^\varepsilon||\tilde{\sigma}_n) \quad (41)$$

Since,

$$D_{\max}(\rho_n^\varepsilon||\tilde{\sigma}_n) = \min\{\alpha : \rho_n^\varepsilon \leq 2^\alpha \tilde{\sigma}_n\},$$

we have,

$$\rho_n^\varepsilon \leq 2^{E_{\max}^\varepsilon(\rho^{\otimes n})} \tilde{\sigma}_n. \quad (42)$$

Using (42) and the operator monotonicity of the logarithm, we infer that

$$S(\rho_n^\varepsilon||\tilde{\sigma}_n) \leq E_{\max}^\varepsilon(\rho^{\otimes n}), \quad (43)$$

since $\text{Tr} \rho_n^\varepsilon \leq \text{Tr} \rho^{\otimes n} = 1$.

From (43) it follows that

$$\limsup_{n \rightarrow \infty} \frac{1}{n} S(\rho_n^\varepsilon||\tilde{\sigma}_n) \leq \mathcal{E}_{\max}^\varepsilon(\rho), \quad (44)$$

and hence,

$$\limsup_{n \rightarrow \infty} \frac{1}{n} E_R(\rho_n^\varepsilon) \leq \mathcal{E}_{\max}^\varepsilon(\rho), \quad (45)$$

where $E_R(\rho_n^\varepsilon) := \min_{\sigma_n \in \mathcal{S}_n} S(\rho_n^\varepsilon||\sigma_n)$. It is known that $E_R(\rho)$ is asymptotically continuous. Hence,

$$\frac{E_R(\rho_n^\varepsilon)}{n} \geq \frac{E_R(\rho^{\otimes n})}{n} - f(\varepsilon), \quad (46)$$

where $f(\varepsilon)$ is a real function of ε satisfying $f(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$. From (45) and (46) we obtain

$$\limsup_{n \rightarrow \infty} \frac{1}{n} E_R(\rho^{\otimes n}) - f(\varepsilon) \leq \mathcal{E}_{\max}^\varepsilon(\rho).$$

Taking the limit $\varepsilon \rightarrow 0$ on both sides of the above inequality yields the desired bound:

$$E_R^\infty(\rho) \leq \mathcal{E}_{\max}(\rho).$$

We next prove the inequality $E_R^\infty(\rho) \geq \mathcal{E}_{\max}(\rho)$.

Consider the sequences $\hat{\rho} = \{\rho^{\otimes n}\}_{n=1}^\infty$ and $\hat{\sigma} = \{\sigma_\rho^{\otimes n}\}_{n=1}^\infty$, where σ_ρ is the separable state for which

$$E_R(\rho) = S(\rho||\sigma_\rho) \equiv \min_{\sigma'} S(\rho||\sigma'). \quad (47)$$

For these two sequences, one can define the following quantity

$$\overline{D}(\hat{\rho}||\hat{\sigma}) := \inf \left\{ \gamma : \limsup_{n \rightarrow \infty} \text{Tr} [\{\rho^{\otimes n} \geq 2^{n\gamma} \sigma_\rho^{\otimes n}\} \rho^{\otimes n}] = 0 \right\}$$

It is referred to as the *sup-spectral divergence rate* and arises in the so-called Information Spectrum Approach [4, 16]. The Quantum Stein's Lemma ([18] or equivalently *Theorem 2* of [16]) tells us that

$$\overline{D}(\hat{\rho}||\hat{\sigma}) = S(\rho||\sigma_\rho) \quad (48)$$

Let us choose

$$\lambda = \overline{D}(\hat{\rho}||\hat{\sigma}) + \delta = E_R(\rho) + \delta, \quad (49)$$

for some arbitrary $\delta > 0$. It then follows from the definition (48) that

$$\limsup_{n \rightarrow \infty} \text{Tr}[\{\rho^{\otimes n} \geq 2^{n\lambda} \sigma_\rho^{\otimes n}\} \rho^{\otimes n}] = 0$$

In particular, for any $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$, such that for all $n \geq n_0$.

$$\text{Tr}[\{\rho^{\otimes n} > 2^{n\lambda} \sigma_\rho^{\otimes n}\} \rho^{\otimes n}] < \frac{\varepsilon^2}{8}. \quad (50)$$

Using Lemma 3 we infer that for all $n \geq n_0$,

$$D_{\max}^\varepsilon(\rho^{\otimes n} || \sigma_\rho^{\otimes n}) \leq n\lambda = nE_R(\rho) + n\delta$$

Hence, $E_{\max}^\varepsilon(\rho^{\otimes n}) \leq nE_R(\rho) + n\delta$, and

$$\mathcal{E}_{\max}^\varepsilon(\rho) \leq E_R(\rho) + \delta.$$

Moreover, since the above bound holds for any arbitrary $\delta > 0$, we deduce that $\mathcal{E}_{\max}^\varepsilon(\rho) \leq E_R(\rho)$. Finally, taking the limit $\varepsilon \rightarrow 0$ on both sides of this inequality yields

$$\mathcal{E}_{\max}(\rho) \leq E_R(\rho). \quad (51)$$

Using the weak additivity (23) of $\mathcal{E}_{\max}(\rho)$, we obtain

$$\begin{aligned} \frac{1}{n} E_R(\rho^{\otimes n}) &\geq \frac{1}{n} \mathcal{E}_{\max}(\rho^{\otimes n}) \\ &= \mathcal{E}_{\max}(\rho). \end{aligned} \quad (52)$$

Taking the limit $n \rightarrow \infty$, on both sides of (52), yields the desired bound

$$E_R^\infty(\rho) \geq \mathcal{E}_{\max}(\rho).$$

■

APPENDIX

Proof of Lemma 2

Proof Define

$$\begin{aligned} \alpha_{AB} &:= 2^\lambda \cdot \sigma_{AB} \\ \beta_{AB} &:= 2^\lambda \cdot \sigma_{AB} + \Delta_{AB}. \end{aligned}$$

and

$$T_{AB} := \alpha_{AB}^{\frac{1}{2}} \beta_{AB}^{-\frac{1}{2}}.$$

Let $|\Psi\rangle = |\Psi\rangle_{ABR}$ be a purification of ρ_{AB} and let $|\Psi'\rangle := T_{AB} \otimes I_R |\Psi\rangle$ and $\rho'_{AB} := \text{Tr}_R(|\Psi'\rangle\langle\Psi'|)$.

Note that

$$\begin{aligned} \rho'_{AB} &= T_{AB} \rho_{AB} T_{AB}^\dagger \\ &\leq T_{AB} \beta_{AB} T_{AB}^\dagger \\ &= \alpha_{AB} = 2^\lambda \cdot \sigma_{AB}, \end{aligned}$$

which implies $D_{\max}(\rho'_{AB} || \sigma_{AB}) \leq \lambda$. It thus remains to be shown that

$$\|\rho_{AB} - \rho'_{AB}\|_1 \leq \sqrt{8 \text{Tr}(\Delta_{AB})}. \quad (53)$$

We first show that the Hermitian operator

$$\bar{T}_{AB} := \frac{1}{2}(T_{AB} + T_{AB}^\dagger).$$

satisfies

$$\bar{T}_{AB} \leq I_{AB}. \quad (54)$$

For any vector $|\phi\rangle = |\phi\rangle_{AB}$,

$$\begin{aligned} \|T_{AB}|\phi\rangle\|^2 &= \langle\phi|T_{AB}^\dagger T_{AB}|\phi\rangle = \langle\phi|\beta_{AB}^{-\frac{1}{2}} \alpha_{AB} \beta_{AB}^{-\frac{1}{2}}|\phi\rangle \\ &\leq \langle\phi|\beta_{AB}^{-\frac{1}{2}} \beta_{AB} \beta_{AB}^{-\frac{1}{2}}|\phi\rangle = \|\phi\|^2 \end{aligned}$$

where the inequality follows from $\alpha_{AB} \leq \beta_{AB}$. Similarly,

$$\begin{aligned} \|T_{AB}^\dagger|\phi\rangle\|^2 &= \langle\phi|T_{AB} T_{AB}^\dagger|\phi\rangle = \langle\phi|\alpha_{AB}^{\frac{1}{2}} \beta_{AB}^{-1} \alpha_{AB}^{\frac{1}{2}}|\phi\rangle \\ &\leq \langle\phi|\alpha_{AB}^{\frac{1}{2}} \alpha_{AB}^{-1} \alpha_{AB}^{\frac{1}{2}}|\phi\rangle = \|\phi\|^2 \end{aligned}$$

where the inequality follows from the fact that $\beta_{AB}^{-1} \leq \alpha_{AB}^{-1}$ which holds because the function $\tau \mapsto -\tau^{-1}$ is operator monotone on $(0, \infty)$ (see Proposition V.1.6 of [3]). We conclude that for any vector $|\phi\rangle$,

$$\begin{aligned} \|\bar{T}_{AB}|\phi\rangle\| &\leq \frac{1}{2} \|T_{AB}|\phi\rangle + T_{AB}^\dagger|\phi\rangle\| \\ &\leq \frac{1}{2} \|T_{AB}|\phi\rangle\| + \frac{1}{2} \|T_{AB}^\dagger|\phi\rangle\| \leq \|\phi\|, \end{aligned}$$

which implies (54).

We now determine the overlap between $|\Psi\rangle$ and $|\Psi'\rangle$,

$$\begin{aligned} \langle\Psi|\Psi'\rangle &= \langle\Psi|T_{AB} \otimes I_R|\Psi\rangle \\ &= \text{Tr}(|\Psi\rangle\langle\Psi|T_{AB} \otimes I_R) = \text{Tr}(\rho_{AB} T_{AB}). \end{aligned}$$

Because ρ_{AB} has trace one, we have

$$\begin{aligned} 1 - |\langle\Psi|\Psi'\rangle| &\leq 1 - \Re\langle\Psi|\Psi'\rangle = \text{Tr}(\rho_{AB}(I_{AB} - \bar{T}_{AB})) \\ &\leq \text{Tr}(\beta_{AB}(I_{AB} - \bar{T}_{AB})) \\ &= \text{Tr}(\beta_{AB}) - \text{Tr}(\alpha_{AB}^{\frac{1}{2}} \beta_{AB}^{\frac{1}{2}}) \\ &\leq \text{Tr}(\beta_{AB}) - \text{Tr}(\alpha_{AB}) = \text{Tr}(\Delta_{AB}). \end{aligned}$$

Here, the second inequality follows from the fact that, because of (54), the operator $I_{AB} - \bar{T}_{AB}$ is positive, and $\rho_{AB} \leq \beta_{AB}$. The last inequality holds because $\alpha_{AB}^{\frac{1}{2}} \leq$

$\beta_{AB}^{\frac{1}{2}}$, which is a consequence of the operator monotonicity of the square root (Proposition V.1.8 of [3]).

Using (2) and the fact that the fidelity between two pure states is given by their overlap, we find

$$\begin{aligned} |||\Psi\rangle\langle\Psi| - |\Psi'\rangle\langle\Psi'| |||_1 &\leq 2\sqrt{2(1 - |\langle\Psi|\Psi'\rangle|)} \\ &\leq 2\sqrt{2\text{Tr}(\Delta_{AB})} \leq \varepsilon. \end{aligned}$$

Inequality (53) then follows because the trace distance can only decrease when taking the partial trace. ■

Proof of Lemma 3

Proof Let Δ_{AB}^+ and Δ_{AB}^- be mutually orthogonal positive operators such that

$$\Delta_{AB}^+ - \Delta_{AB}^- = \rho_{AB} - 2^\lambda \sigma_{AB}.$$

Furthermore, let P_{AB} be the projector onto the support of Δ_{AB}^+ , i.e.,

$$P_{AB} = \{\rho_{AB} > 2^\lambda \sigma_{AB}\}.$$

We then have

$$\begin{aligned} P_{AB}\rho_{AB}P_{AB} &= P_{AB}(\Delta_{AB}^+ + 2^\lambda \sigma_{AB} - \Delta_{AB}^-)P_{AB} \\ &\geq \Delta_{AB}^+ \end{aligned}$$

and, hence,

$$\sqrt{8\text{Tr}(\Delta_{AB}^+)} \leq \sqrt{8\text{Tr}(P_{AB}\rho_{AB})} = \varepsilon.$$

The assertion now follows from Lemma 2 because

$$\rho_{AB} \leq 2^\lambda \sigma_{AB} + \Delta_{AB}^+.$$

■

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 - [33] A full entanglement monotone or an LOCC monotone, is one which does not increase *on average* under local operations and classical communications (LOCC), i.e., if an LOCC map on a quantum state ρ results in a state ρ_i with probability p_i , then $E_{\max}(\rho) \geq \sum_i p_i E_{\max}(\rho_i)$.
 - [34] An entanglement monotone $E(\rho)$ is weakly additive if $E(\rho^{\otimes n}) = nE(\rho)$ for any positive integer n .
 - [35] This can be proved by showing that $E_{\min}(\rho)$ violates the requirement (Lemma 1 of [31]) that any entanglement monotone, when evaluated on a pure state, should be a concave function of its partial trace. For a pure state $\rho = |\psi\rangle\langle\psi| \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$, $E_{\min}(\rho) = -\log \lambda_{\max}(\rho_{\psi}^A)$, where $\rho_{\psi}^A = \text{Tr}_B |\psi\rangle\langle\psi|$, and λ_{\max} denotes its maximum eigenvalue. Since the maximum eigenvalue of a density matrix is a convex function, as is the negative of the logarithm, $E_{\min}(\rho)$ is a *convex* function of ρ_{ψ}^A .
 - [36] We would like to point out that in [7] a different modification of the log robustness was shown to equal $E_R^{\infty}(\rho)$, by a method different to the one employed in this paper. The main difference in the modification was that in [7], the smoothing parameter depended on the number n of copies of the given state ρ .
 - [37] Note, however, that this bound is loose for bipartite states which are close to separable states. For an operational significance of the robustness in relation to entanglement activation see [5].